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# Design of Reliable Control Systems Possessing Actuator Redundancies

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**The explicit definitions of actuator redundancies are introduced. A design scheme is then presented to synthesize reliable control against actuator failures for dynamic systems possessing actuator redundancies. The properly designed controller is able to guarantee the stability and to maintain the steady-state tracking performance in the event of actuator failures. The design method is based on a robust regional eigenvalue assignment technique with the help of a precompensator. The effectiveness of the proposed method has been verified using an example of the aircraft bank angle control problem.**

## I. Introduction

METHODOLOGY used in either classical or modern control system design often presumes that all system components are in good working conditions. As a result, a majority of control systems designed using conventional techniques may not be able to maintain a satisfactory performance in the presence of component failures. In some cases, even the closed-loop system stability may be in jeopardy. However, if there exist some redundancies in the system, it will often be possible to design controllers such that failures in some system components will not cause immediate threats to the safety of the overall system. This will give human operators additional time to carry out necessary safety assurance steps before catastrophe strikes. Such design methodologies are particularly important in safety-critical systems, such as space missions, aircraft, nuclear or chemical reactors, etc.[1–3] A control system designed to tolerate failures in system components, while maintaining an acceptable closed-loop system stability/performance, has been defined as a fault-tolerant control system.[4]

There are two main design philosophies for fault-tolerant control, active and passive fault-tolerant control. They rely heavily on existing system redundancies to achieve tolerable performance degradation in the event of component failures. In fact, the design of a fault-tolerant control system can be viewed as the management of system redundancy to increase the system reliability.[5] In active fault-tolerant control systems, the design involves such procedures as real-time fault detection, isolation, and control system reconfiguration. The redundancy in such systems can be in an analytic form, for example, the mathematical model of the system. This kind of redundancy is usually referred to as analytic redundancy.[6] The controller with such a design philosophy is referred to as reconfigurable control in the literature.[7–9]

Passive fault-tolerant control is referred to as reliable control in the literature.[10] In a reliable control system, the controller is designed to be robust to a presumed set of component faults. The fault scenarios have already been taken into account at the controller design stage. Once the design is completed, the controller will remain fixed. The design procedure for such controllers is often more involved because the control system has to perform satisfactorily not only during the normal system operation but also under various component failures. It should be emphasized that the redundancies in such systems are usually in hardware forms, such as redundant sensors, actuators, or controllers. Consequently, this class of systems are often represented by multivariable system models.

The original concept of redundancy can be traced back to the 1950s when von Neumann proposed the idea of building a reliable computer. Inspired by such a novel idea, a multiple control system

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structure was proposed by Siljak in the context of decentralized control for complex systems.[11,12] The essential concept of redundancy was introduced in the form of parallel connection of multiple controllers. Note that the idea of redundancy in Siljak's work is explicitly used in a control system framework for the first time. By the use of the multiple control system structure, faults/failures in some of the controllers can be tolerated, and the overall system reliability can be improved. Similarly, reliable control design using multiple/redundant controllers has been extensively studied, and many results have been obtained.[13–15]

In the literature, the reliable control design problem is also studied in the framework of control systems possessing integrity. In multivariable systems, the interaction among different loops makes the control systems design more difficult, and when there are failures in some loops, the interaction among the remaining loops may result in the reduction of the system stability margins. Such a problem brought up the question of system integrity.[16] A multivariable feedback system is said to have high integrity if it remains stable under all likely failures. Here, all likely failures mean all types of loop failures, such as output transducer, controller, and actuator failures. In 1985, Shimemura and Fujita formulated this problem using the state-space representation for actuator failure cases, and a state feedback design scheme was proposed based on a Riccati-type equation.[17] Similarly, a sequential design scheme has been developed in Ref. 18 based on the solution of matrix Lyapunov equations. In the cited works, only the stabilization of open-loop stable systems has been considered and may not be applicable to open-loop unstable systems. The design problem for control systems possessing integrity has further been formulated in an $H_\infty$ framework, where the performance of the system in the event of component failures can be maintained in an $H_\infty$ optimal sense.[10,19] The synthesis of such reliable control systems has also been investigated as a simultaneous stabilization problem using the method of stable factorization.[20,21]

In the mentioned works, the control freedom of multivariable systems has been utilized effectively to design reliable control systems, but without explicit examination of the system redundancy. Note that, for any reliable control system, hardware redundancy is the key ingredient, and a good reliable control design should have full access to and utilization of such redundancies. Based on this point of view, a design method for reliable control systems against actuator failures has been proposed in Ref. 22, where the actuator redundancies have been clearly illustrated for multiple-input/single-output systems. However, only system stability has been considered therein; the tracking ability cannot be maintained under actuator failures.

In this paper, the reliable control problem against actuator failures is further investigated. Unlike the design problem of reliable control with multiple controllers, it is assumed that the controllers are functional. What we are interested in is how to design a controller that can stabilize the system even in the presence of actuator failures. First, the analysis of actuator redundancy has been examined in depth, and explicit definitions of actuator redundancy have been illustrated for multi-input/multi-output (MIMO) systems. The proposed design makes use of system actuator redundancy and guarantees both stability and tracking ability even when some actuators fail. A dynamic precompensator has been used to modify the dynamics of the actuator channels and has played an important role in the overall design. A robust regional eigenvalue assignment technique has been modified to synthesize the state feedback gains so that any combinations of actuator failures will not cause any closed-loop eigenvalues to migrate outside of the predefined stability region. In addition, steady-state tracking ability can be achieved under all fault scenarios through a *PI* controller. Note that the proposed design is only for multiple-input/single-output (MISO) systems. For MIMO systems, the relationship between inputs and outputs are usually complex. Although actuator redundancies are defined and examined for general MIMO systems in this paper, how to utilize such actuator redundancy and design reliable control are difficult and still an open problem.

The unique feature of the paper is that it highlights the importance of the redundancies in the design of reliable control systems and presents a systematic synthesis procedure with clear physical insight in each step. The design method can be used not only for open-loop stable systems, but also for open-loop unstable ones. The paper is organized as follows: Section II gives definitions of actuator redundancies. The procedure in synthesizing the precompensator and the reliable controller is introduced in Sec. III. A design example is presented in Sec. IV, followed by the Conclusions in Sec. V.

## II. Actuator Redundancy in Control Systems

Redundancy is the basic ingredient in any fault-tolerant system. It simply refers to the additional system resources, such as sensors, actuators, or controllers, that are beyond what is needed to achieve the intended control objectives during the normal system operation. The existence of redundancy is solely for combating faults in the system.

In practice, actuators are usually large in size and require a significant amount of power to operate. It is usually very difficult to duplicate actuators at a control point. In fact, the actuator redundancies can only be introduced by means of additional manipulated variables in the system, which have independent control over system outputs. These redundant system variables may take totally different physical forms or meanings. In other words, the dynamics from each actuator to the system output may be different. When we consider actuator redundancy, it is important to differentiate actuator inputs from system inputs. Manipulated variables are the physical variables manipulated by actuator inputs. Because the redundant actuators have to be in operation at all times, the controller design for such systems becomes more complex.

Consider a controllable and observable linear time-invariant system with $p$ actuators and $m$ output. The system can be represented as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{y} = C\boldsymbol{x} \tag{1}$$

where $\boldsymbol{x} \in \Re^{n\times 1}$, $\boldsymbol{u} \in \Re^{p\times 1}$, and $\boldsymbol{y} \in \Re^{m\times 1}$ are the system state, the input vector, and the output vector, respectively; correspondingly, the state transition and the output matrices, $A$ and $C$, have appropriate dimensions. The system input matrix $B \in \Re^{n\times p}$ can be represented by $B = [\boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \cdots\ \ \boldsymbol{b}_p]$ with each column being $\boldsymbol{b}_i \in \Re^{n\times 1}$, $1 \le i \le p$. The system input vector associated with the multiple actuators can be written as $\boldsymbol{u} = [\boldsymbol{u}_1\ \ \boldsymbol{u}_2\ \ \cdots\ \ \boldsymbol{u}_p]^T$.

For a dynamic system, more often than not, state controllability alone is not sufficient for certain control objectives, such as regulator and tracking design. Another controllability criterion describing the relationships between the system inputs and outputs, known as functional controllability,[23,24] proves to be useful. Such a concept can also be utilized to define the actuator redundancies.

*Definition 1 (Functional Controllability)*[23]*:* A system is said to be functionally controllable if for any given suitable output vector $\boldsymbol{y}(t)$ (defined for $t \ge 0$) there exists an input vector $\boldsymbol{u}(t)$ (defined for $t \ge 0$) for which this output vector $\boldsymbol{y}(t)$ can be achieved from the zero initial condition.

Here, suitable means that the output vector is sufficiently smooth to be generated without impulse functions in $\boldsymbol{u}$, and has a Laplace transform. This definition, in fact, guarantees the independent control of the system inputs over the system outputs. To satisfy the functional controllability, the number of inputs must be greater than or equal to the number of outputs, that is, $p \ge m$. A necessary and sufficient condition for the functional controllability is illustrated in the following lemma.[23,24]

*Lemma 1:*

1) For $p = m$, the system is said to be functionally controllable if and only if the determinant of its transfer function matrix $G(s) = \{\boldsymbol{c}_j(Is - A)^{-1}\boldsymbol{b}_i,\ 1 \le i \le p;\ 1 \le j \le m\}$ is nonzero, that is, $|G(s)| \ne 0$.

2) For $p > m$, the system is said to be functionally controllable if and only if there exists at least one nonzero $m \times m$ minor in the transfer function matrix $G(s) = \{\boldsymbol{c}_j(Is - A)^{-1}\boldsymbol{b}_i,\ 1 \le i \le p;\ 1 \le j \le m\}$.

Based on the Definition 1 and Lemma 1, the actuator redundancy can be examined. Because for each actuator there is an associated manipulated variable in the system, different manipulated variables often affect the system differently unless there exists some degree of symmetry in the system. Therefore, actuators in different part of the system can take totally different physical forms. In general, one can classify the actuator redundancy into two types: 1) nonuniform actuator redundancy and 2) uniform actuator redundancy.

*Definition 2 (Nonuniform Actuator Redundancy):* The system described in Eq. (1) is said to possess $(p - m)$ degree of nonuniform actuator redundancy if 1) the system is completely state controllable; 2) the number of inputs is greater than the number of outputs, that is, $p > m$; and 3) rank $[B] > m$ and there exist $C_p^m = p!/(p-m)!m!$ nonzero $m \times m$ minors, or in other words, $C_p^m$ nonsingular $m \times m$ square submatrices in the system transfer function matrix $G(s)$.

In the Definition 2, the first condition guarantees that the system is always stabilizable using the state feedback. The second and third conditions are simply extensions of the functional controllability in the context of actuator redundancy. They guarantee that the system is functionally controllable by any $m$ inputs out of the total $p$ inputs, which simply means that the system has $(p - m)$ degree of actuator redundancy.

This concept can further be interpreted as follows.

For a given $m \times p$ transfer function matrix $G(s)$, let us define two index sets: $I = \{1, \ldots, m\}$ and $J = \{j_1, j_2, \ldots, j_m\} \subset \{1, \ldots, p\}$. We can form a nonsingular $m \times m$ submatrice out of the transfer function matrix, denoting it as $G_{(I,J)}(s)$. Let its complement be $G_{(I,\bar{J})}(s)$. Clearly, $G_{(I,\bar{J})}(s)$ has a dimension of $m \times (p - m)$. If we partition the system input vector accordingly, the system output can be represented as

$$Y(s) = G_{(I,J)}(s) \cdot U_{(J)}(s) + G_{(I,\bar{J})}(s) \cdot U_{(\bar{J})}(s) \tag{2}$$

where $U_{(J)}(s)$ and $U_{(\bar{J})}(s)$ are the Laplace transform of the system input vector partitioned based on the index sets $J$ and $\bar{J}$.

Clearly, the system output is composed of two parts: The first part is composed of a group of $m$ inputs, and the second part is the remaining $(p - m)$ inputs. In the absence of actuator failures, the first $m$ inputs alone can maintain the functional controllability of the system. In the presence of an actuator failure within the first group of $m$ inputs, if any one of the remaining $(p - m)$ inputs can fill in for the failed actuator, this actuator would be classified as redundant. Therefore, the system is said to possess $(p - m)$ degree of actuator redundancy if for any combination of $m$ inputs, that is, for any $J = \{j_1, j_2, \ldots, j_m\} \subset \{1, \ldots, p\}$, $p \geq m$, $G_{(I,J)}(s)$ is non-singular or the system transfer function matrix $G(s)$ has $C_p^m = p!/(p-m)!m!$ nonzero $m \times m$ minors.

This definition deals with a situation where any actuator input can be regarded as a redundant input with respect to any system output. Usually these actuators act on the system independently from different locations in the system. The control signals passing through these actuator channels often affect system outputs independently. This is a general characteristic of a physical system with dissimilar actuators.

For some practical systems, particularly large systems, it may be possible that a certain input can only be considered as a redundant input to a subset of the system outputs. In other words, certain system inputs may be decoupled from certain system outputs. In this case, the actuators associated with the decoupled system inputs can no longer be counted as the redundant actuators for those system outputs. From a mathematical point of view, this situation corresponds to the case where the number of nonzero $m \times m$ minors is less than $C_p^m$, but greater than one.

Should such a situation arise, it is necessary to subdivide the system and the corresponding inputs first. Within these subsystems, the redundant actuators can be identified. One can also think of this problem as pairing a group of system inputs with a group of system outputs with the number of inputs greater than that of the outputs. There exists one extreme case, and the actuator redundancy in this case can be called uniform redundancy or dependant redundancy.

*Definition 3 (Uniform Actuator Redundancy):* The system described in Eq. (1) is said to possess $(p - m)$ degree of uniform actuator redundancy if 1) the system is completely state controllable; 2) the number of inputs is greater than or equal to the number of outputs, that is, $p \geq m$; 3) the input matrix $[B]$ satisfies rank $[B] = m$; and 4) there exist $\gamma$ nonzero $m \times m$ minors in the system transfer function matrix $G(s)$, where $1 \leq \gamma \leq C_p^m$.

That the rank of the $B$ matrix is equal to the number of outputs implies that the dynamics from some actuator inputs to the system states are similar (maybe different by some constant scaling factors). In other words, this condition corresponds to a physical situation

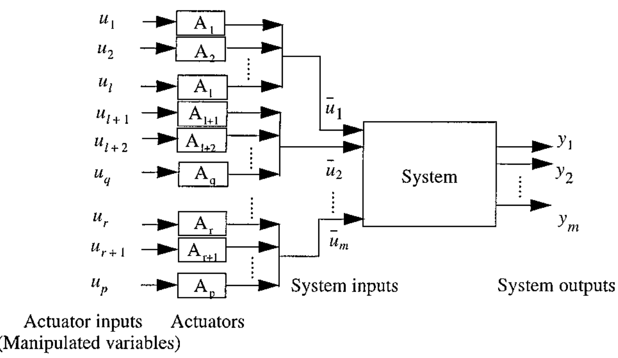



**Fig. 1 Interpretation of the uniform actuator redundancy.**

where some actuators are associated with a single system input and act on the system at the same input location (or at the symmetric points for symmetrical systems). In this case, overall $p$ actuators actually act onto the system at $m$ different locations. This concept can be seen clearly from Fig. 1.

Experience has shown that design of reliable control becomes much easier if the actuator redundancy is of the uniform type. Note that this type of redundancy exists only in very ideal situations; in practice, few systems possess such a nice property. Fortunately, for most practical systems, we can convert the nonuniform actuator redundancy to a uniform one by using a properly designed dynamic precompensator.

## III. Synthesis of Reliable Control Systems

### A. Model of Actuator Failures and Structure of Reliable Control Systems

Because only single system output is considered in the design, that is, $m = 1$, the system can be represented in a MISO form, which has $p - 1$ degree of actuator redundancy. The basic requirement of reliable controller design for such a system is that the control system has to utilize fully the actuator redundancy and to perform satisfactorily not only when all actuators are in good working condition, but also when some have failed.

Consider the dynamic system given in Eq. (1). Let us define a diagonal matrix

$$L = \mathrm{diag}(l_1, l_2, \ldots, l_p) \tag{3}$$

where $L$ is a $p \times p$ identity matrix if all actuators are in good working condition. A failure in the $i$th actuator is modeled by setting the $i$th diagonal element in $L$ to zero, that is, $l_i = 0$. By the denoting of $\boldsymbol{u}_a(t)$ as the actuator output signal, it can be represented as follows:

$$\boldsymbol{u}_a(t) = L \cdot \boldsymbol{u}(t) + (I - L) \cdot \boldsymbol{f}_a \tag{4}$$

where $\boldsymbol{f}_a$ is a constant offset in the actuator output caused by the failure of the actuator. It corresponds to the value at which the actuator is stuck. From the preceding representation, the system with possible actuator failures can be represented in the state-space form by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}_a(t) = A\boldsymbol{x}(t) + BL\boldsymbol{u}(t) + B(I - L)\boldsymbol{f}_a$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{5}$$

In general, because different actuators affect the system from different input locations, the rank of the $B$ matrix is usually not unity for single-output systems. In other words, the system has nonuniform actuator redundancy.

The system in Eq. (5) can also be represented in transfer function form as

$$Y(s) = G_f(s)U(s) + F(s) \tag{6}$$

where

$$G_f(s) = G(s)L = \frac{\boldsymbol{N}(s)L}{d_0(s)} = \frac{1}{d_0(s)}[n_1(s) \quad n_2(s) \quad \cdots \quad n_p(s)]L$$

$$F(s) = \frac{\boldsymbol{N}(s)(I - L)}{d_0(s)s}\boldsymbol{f}_a \tag{7}$$

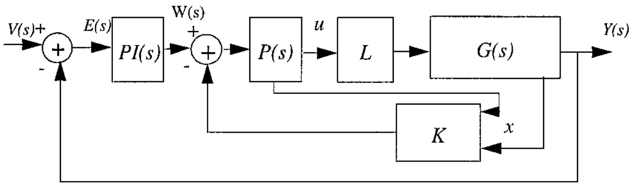



**Fig. 2 Reliable control system with a dynamic precompensator.**

where $G(s)$ is the nominal system and $G_f(s)$ is the system with actuator failures. The $n$th-order polynomial $d_0(s)$ is the characteristic polynomial of the system, and $\boldsymbol{N}(s)$ is the numerator polynomial vector of the dimension $1 \times p$. $F(s)$ corresponds to the output bias caused by the actuator failures.

Note that there are $2^p$ possible actuator failure modes in total represented by the combination of the elements in $L$. However, we will only consider $2^p - 1$ modes, excluding the case where $L$ is a zero matrix, that is, there should at least be one healthy actuator available in the system. This requirement is essential for the proposed scheme to work with open-loop unstable systems.

Based on the given formulation, the main objective of the reliable control system design is to synthesize a fixed controller so that the stability and the performance of the closed-loop system are maintained so long as there is one healthy actuator left in the system.

The structure of the overall system is shown in Fig. 2. The overall control system consists of 1) a dynamic precompensator $P(s)$, 2) a state feedback controller $K$, and 3) a *PI* controller $PI(s)$.

There are two main steps involved in the design process:

1) Convert the nonuniform actuator redundancy to a uniform one by using a properly designed dynamic precompensator.

2) Select a state feedback as well as $PI$ controller gains using a robust regional eigenvalue assignment technique.

Note that the *PI* regulator is used mainly to eliminate the bias caused by the malfunctioned actuator (stuck faults) to achieve the desired steady-state performance. However, we have to make sure that the good steady-state performance is not achieved at the expense of the transient performance of the system. Therefore, for a satisfactory performance in both transient and steadystate, the *PI* controller and the state feedback controllers should be designed simultaneously.

### B. Design of the Precompensator

For the system given in Eq. (1), a precompensator with the following diagonal matrix form is used:

$$P(s) = [1/d_c(s)]\mathrm{diag}[m_1(s) \quad m_2(s) \quad \cdots \quad m_p(s)] = [1/d_c(s)]\boldsymbol{M}(s) \tag{8}$$

where $d_c(s)$ is a Hurwitz polynomial chosen such that every element in $P(s)$ is proper.

The choice of the numerator polynomials $m_i(s)$ depends on the numerator polynomial vector of the nominal system, that is, $\boldsymbol{N}(s)$. From $\boldsymbol{N}(s) = [n_1(s) \ n_2(s) \ \cdots \ n_p(s)]$, the least common polynomials of $n_i(s)/b_i$, $i = 1, 2, \ldots, p$, can be obtained, where $h_1, h_2, \ldots, h_p$ are the coefficients of the highest power terms in $n_i(s)$, $i = 1, 2, \ldots, p$, respectively. Let this least common polynomial be denoted by

$$n_c(s) = s^r + b_{r-1}s^{r-1} + \cdots + b_0 \tag{9}$$

The elements of the numerator polynomial vector of the precompensator, $b_1(s), b_2(s), \ldots, b_p(s)$ can then be determined as follows:

$$m_1(s) = \frac{n_c(s)}{n_1(s)/h_1}$$

$$m_2(s) = \frac{n_c(s)}{n_2(s)/h_2}, \ldots, m_p(s) = \frac{n_c(s)}{n_p(s)/h_p} \tag{10}$$

With such a precompensator, the augmented system becomes

$$G_A(s) = G_f(s)P(s) = \frac{\boldsymbol{N}(s)L\boldsymbol{M}(s)}{d_0(s)d_c(s)} = \frac{\boldsymbol{N}(s)\boldsymbol{M}(s)L}{d_0(s)d_c(s)} \tag{11}$$

Note that because the matrices $M(s)$ and $L$ are both diagonal, their positions can be interchanged. From Eq. (10), one can obtain that

$$\begin{aligned}\boldsymbol{N}(s)\boldsymbol{M}(s) &= [n_1(s) \quad n_2(s) \quad \cdots \quad n_p(s)] \\ &\quad \times \begin{bmatrix} m_1(s) & 0 & \cdots & 0 \\ 0 & m_2(s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & m_p(s) \end{bmatrix} \\ &= [h_1 n_c(s) \quad h_2 n_c(s) \quad \cdots \quad h_p n_c(s)] \\ &= n_c(s)[h_1 \quad h_2 \quad \cdots \quad h_p]\end{aligned} \tag{12}$$

Letting $\boldsymbol{H} = [h_1 \ h_2 \ \cdots \ h_p] \in \Re^{1\times p}$, we can further write the polynomial $\boldsymbol{N}(s)\boldsymbol{M}(s)$ as

$$\boldsymbol{N}(s)\boldsymbol{M}(s) = n_c(s)\boldsymbol{H} = \left(s^r + b_{r-1}s^{r-1} + \cdots + b_0\right)\boldsymbol{H} \tag{13}$$

Note that Eq. (13) basically implies that every actuator channel in the compensated system has the same dynamics and may differ only by constant gains as specified in the vector $\boldsymbol{H}$.

Without any loss of generality, let us assume that the denominator polynomial of the compensated system has the following form:

$$D(s) = d_0(s)d_c(s) = s^q + d_{q-1}s^{q-1} + d_{q-2}s^{q-2} + \cdots + d_1 s + d_0 \tag{14}$$

The order of the compensated system, $q$, has now become the sum of the order of the nominal system and that of the precompensator.

With appropriate choice of the state variables, the transfer function matrix in Eq. (11) can be represented in the following state-space form:

$$\begin{aligned}\dot{\boldsymbol{x}}_a(t) &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{q-1} \end{bmatrix}\boldsymbol{x}_a(t) \\ &\quad + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ b_1 & b_2 & b_3 & \cdots & b_m \end{bmatrix}\boldsymbol{u}_a(t) \\ &= A_a\boldsymbol{x}_a(t) + B_a L\boldsymbol{u}(t) + B_a(I - L)\boldsymbol{f}_a\end{aligned} \tag{15}$$

$$\boldsymbol{y}(t) = [b_0 \quad b_1 \quad \cdots \quad b_{r-1} \quad 1 \quad 0 \quad \cdots \quad 0]\boldsymbol{x}_a(t) = C_a\boldsymbol{x}_a(t) \tag{16}$$

From Definition 3, it is clear that Eq. (15) represents a system with uniform actuator redundancy.

Note that the dynamic precompensator is implemented as a part of the controller, and it plays an important role in the overall reliable control system design. The use of the precompensator converts the nonuniform actuator redundancy to the uniform one, and this can significantly simplify the controller gain selection process for the later stage.

From the selection procedure of the precompensator, it can be seen that there exists some degree of freedom in choosing the denominator $d_c(s)$. The following guidelines are proved to be useful.

1) The order of $d_c(s)$ should at least be equal to the highest order of the numerator polynomial of the precompensator $m_i(s)$ for causality; therefore, the minimal order of the precompensator depends entirely on the numerator order of the system model. In order not to introduce slow dynamics to the closed-loop system, it is recommended that the order of $d_c(s)$ is equal to the highest order of the numerator polynomial of the precompensator, $m_i(s)$.

2) To maintain the controllability of the system, $d_c(s)$ should not have any common roots with the numerator polynomial vector of the nominal system, $N(s)$.

Furthermore, such freedom in choosing $d_c(s)$ is proved to be helpful in the synthesis of the feedback gain in the later stage. In fact, the concept of employing precompensators in the reliable control system design is not new. It has been shown in Ref. 25 that one of the advantages of using a dynamic precompensator is that the degree of freedom of reliable control system design can be increased significantly. Consequently, solutions to reliable control system design can be found for systems where there were no solutions before.

### C. Design of Reliable State Feedback and *PI* Controller

As can be seen from Fig. 2, in addition to the precompensator, the inner loop of the system also consists of a state feedback controller. Let us assume that the state feedback gain matrix takes the following form:

$$K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1q} \\ k_{21} & k_{22} & \cdots & k_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ k_{p1} & k_{p2} & \cdots & k_{pq} \end{bmatrix} \tag{17}$$

With the state feedback, the inner-loop transfer function from $W(s)$ to $Y(s)$ can be expressed as

$$Y(s) = \frac{n_c(s)HL}{Q(s)}W(s) + \frac{n_c(s)H(I - L)}{Q(s)s}f_a \tag{18}$$

where the denominator polynomial can be obtained from Eq. (15) as follows:

$$\begin{aligned} Q(s) &= \det[sI - (A_a - B_a LK)] \\ &= s^q + \left(d_{q-1} + \sum_{i=1}^{p} b_i l_i k_{iq}\right)s^{q-1} + \cdots + \left(d_0 + \sum_{i=1}^{p} b_i l_i k_{i1}\right) \end{aligned} \tag{19}$$

Notice from Eq. (19) that each column of matrix $K$ affects only one coefficient in the preceding characteristic equation. This feature is a direct result of uniform redundancy, which makes the synthesis of the controller straightforward.

In the outer loop, a *PI* controller is used to achieve a desired steady-state performance in the presence of actuator failures. A more detailed structure of the overall system is shown in Fig. 3.

By some simple block diagram manipulation, the transfer function of the closed-loop system can be obtained as

$$Y(s) = \frac{n_c(s)PI(s)\sum_{i=1}^{p} h_i l_i}{Q(s) + n_c(s)PI(s)\sum_{i=1}^{p} h_i l_i}V(s) + D_f(s)$$

$$D_f(s) = \frac{n_c(s)\sum_{i=1}^{p} h_i(1 - l_i)f_a}{sQ(s) + n_c(s)PI(s)\sum_{i=1}^{p} h_i l_i} \tag{20}$$

where $V(s)$ is the closed-loop system input. The *PI* regulator has the following form:

$$PI(s) = k_P + (k_I/s) \tag{21}$$

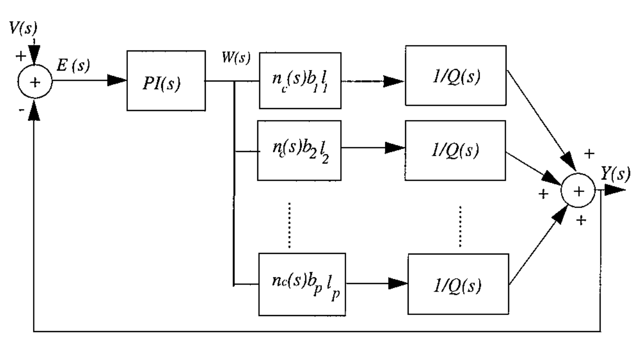



**Fig. 3 Block diagram of the system.**

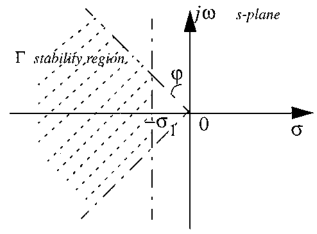



**Fig. 4 Γ-stability region on s-plane.**

where $\{k_P, k_I\}$ are the proportional and integral gains of the *PI* regulator.

By substituting Eq. (21) into Eq. (20), it is clear that

$$\begin{aligned} d_f(\infty) &= \lim_{s\to 0} s \cdot D_f(s) \\ &= \lim_{s\to 0} \frac{sn_c(s)\sum_{i=1}^{p} h_i(1 - l_i)f_a}{sQ(s) + sk_p n_c(s)\sum_{i=1}^{p} h_i l_i + k_I n_c(s)\sum_{i=1}^{p} h_i l_i} = 0 \end{aligned} \tag{22}$$

Therefore, the effect of the actuator failure in steady state can be eliminated by the *PI* controller.

The characteristic equation of the closed-loop system [Eq. (20)] can be written in the following form by using Eq. (21):

$$\begin{aligned} Q_c(s) &= sQ(s) + sk_p n_c(s)\sum_{i=1}^{p} h_i l_i + k_I n_c(s)\sum_{i=1}^{p} h_i l_i \\ &= s^{q+1} + d'_q s^q + d'_{q-1}s^{q-1} + \cdots + d'_0 \end{aligned} \tag{23}$$

It can be seen clearly that Eq. (23) allows us to design both the state feedback controller and the *PI* regulator simultaneously to achieve the desired transient and steady-state performance. Hence, the problem of the reliable control system design is now to synthesize a feedback gain matrix $K$ and a *PI* regulator, so that, for any combination of actuator failure modes, the roots of the characteristic equation (23) should remain inside the desired stability region in the $s$ plane.

A commonly used stability region in the $s$ plane for an acceptable transient performance is known as the Γ-stability region, as shown in Fig. 4.[25–27] This region may degenerate to an open half-plane on the left of a straight line at $\sigma = -\sigma_1$. The eigenvalues in this open half-plane will guarantee the stability of the closed-loop system with a stability margin of at least $\sigma_1$.

This would have been a standard eigenvalue assignment problem if $L$ had always been an identity matrix. However, in this case, there are $2^p - 1$ different combinations of $L$ that have to be taken into consideration. Therefore, we are unable to specify the exact locations for the eigenvalues of the closed-loop system, but only a desired region.

Although there is extensive research work done on the robust stability of a polynomial subject to parameter perturbations, which are based on Kharitonov's theorem and its various generalizations,[28] unfortunately, they are not suited for our problem. These works deal mainly with the robustness analysis instead of controller synthesis. Although the robust control problem can be clearly defined and well posed from a mathematical point of view, there are no known simple, necessary, and sufficient conditions for the existence of a robustly stabilizing controller.[25] Furthermore, the control problem proposed here is even harder because we are dealing with actuator failures that often manifest themselves as large changes in the coefficients of the characteristic polynomial rather than parameter perturbations.

To calculate the feedback gain matrix $K$ and the *PI* controller parameters from the coefficients of the characteristic equation (23), the first step is to convert the desired stability region in the $s$ plane into the corresponding interval polynomial in the parametric space.[25,27,29] In 1991, Soh et al.[30] proposed a numerical algorithm to evaluate the maximal upper and lower bounds on the coefficients of a given polynomial such that the roots of the interval polynomial are within the specified region in the $s$ plane (see the Appendix). However, this scheme can not be used directly for our problem because the obtained bounds are usually very small and conservative.

Instead of using a single nominal polynomial [$P_0(s)$ in Eq. (A2), see Appendix] to calculate the coefficient intervals, our solution is to use a set of nominal polynomials to obtain their coefficient intervals. A suitable choice of such a polynomial set is the following $(q+1)$th-order binomial polynomial with an intermediate parameter $\sigma$; the coefficients are varied by adjusting $\sigma$. This nominal polynomial set is called the reference polynomial set,

$$\Psi(s,\sigma) = (s+\sigma)^{q+1} = s^{q+1} + \alpha_q(\sigma)s^q + \alpha_{q-1}(\sigma)s^{q-1} + \cdots + \alpha_0(\sigma) \qquad \sigma > 0 \tag{24}$$

For every chosen $\sigma$, the coefficients intervals of the preceding polynomial can be determined. This will generate an interval polynomial set, whose coefficient bounds are also functions of the intermediate parameter $\sigma$,

$$\Phi(s,\sigma) = s^{q+1} + [\underline{\alpha}_q(\sigma), \bar{\alpha}_q(\sigma)]s^q + [\underline{\alpha}_{q-1}(\sigma), \bar{\alpha}_{q-1}(\sigma)]s^{q-1} + \cdots + [\underline{\alpha}_0(\sigma), \bar{\alpha}_0(\sigma)] \tag{25}$$

From the interval polynomial set, a so-called $Q$-box set can be introduced as

$$Q = \{\Omega(\sigma) \mid \alpha_i(\sigma) \in [\underline{\alpha}_i(\sigma), \bar{\alpha}_i(\sigma)], i = 0, 1, \ldots, q\} \tag{26}$$

where $\underline{\alpha}_i(\sigma) = \alpha_i(\sigma) - \tau_i\varepsilon$, $\bar{\alpha}_i(\sigma) = \alpha_i(\sigma) + \mu_i\varepsilon$. Here, $\varepsilon > 0$ is the size of the perturbation used in the bound computation and $\tau_i$ and $\mu_i$ are the corresponding weights on each perturbation [see Appendix Eq. (A1)]. It is clear that this $Q$-box set represents a set of stable interval polynomials. With appropriate choice of the weights, most of the practical stability regions, such as the $\Gamma$ stability region, can be dealt with. As can be seen clearly, by using the $Q$-box set, the admissible coefficient space for guaranteeing the stability is enlarged significantly. It will no longer be restricted to one set of coefficient bounds that is based on one nominal polynomial; instead, a number of overlapped coefficient bounds can be obtained by varying $\sigma$ continuously as shown in Eq. (26).

Although the preceding reference polynomial set [Eq. (24)] is surprisingly simple, it has several desirable properties: 1) there is only a single intermediate parameter that needs to be adjusted, which makes the coefficient bounds calculation straightforward, and 2) the stability margin of the system associated with this polynomial set is simply the parameter $\sigma$, which makes the relationship between the relative stability and the coefficient bounds transparent.

Once the interval polynomial set is obtained, the feedback gain matrix and the *PI* controller parameters can be chosen so that the coefficients of Eq. (23), subject to all actuator failure modes, are within the respective bounds of any polynomial in the polynomial set Eq. (25), or in other words, within the robustly stable $Q$-box set. To achieve this objective, a system of inequalities needs to be solved. They are shown as follows:

$$\underline{\alpha}_q(\sigma_i) < d'_q(d_{q-1}, K, L_i, k_P, k_I) < \bar{\alpha}_q(\sigma_i)$$

$$\underline{\alpha}_{q-1}(\sigma_i) < d'_{q-1}(d_{q-2}, K, L_i, k_P, k_I) < \bar{\alpha}_{q-1}(\sigma_i)$$

$$\cdots\cdots$$

$$\underline{\alpha}_0(\sigma_i) < d'_0(K, L_i, k_P, k_I) < \bar{\alpha}_0(\sigma_i)$$

$$i = 1, 2, \ldots, 2^p - 1 \tag{27}$$

where $d'_j(d_{j-1}, K, L_i, k_P, k_I)$, $j = 0, 1, \ldots, q$, are the coefficients of $s^j$ in Eq. (23). $L_i$ is the actuator failure representation matrix with the $i$th combination of the diagonal elements being zero or unity Eq. (3), and $\underline{\alpha}_j(\sigma_i)$ and $\bar{\alpha}_j(\sigma_i)$, $j = 0, 1, \ldots, q$, are the upper and lower bounds as in Eq. (25) with respect to $L_i$. Note that the denominator coefficients of the precompensator are related to $d'_j(\cdot)$ through $d_{j-1}$ in Eq. (14). By solving the preceding inequalities using a numerical algorithm, an admissible feedback gain matrix and the *PI* controller parameters can be obtained.

This design problem can also be visualized by the following geometric interpretation. A two-dimensional parametric space and corresponding map on the $s$ plane is shown in Fig. 5. Because the intervals in Eq. (25) are functions of $\sigma$, the size of the intervals for each coefficient will change as $\sigma$ varies. Note from the numerical results that as $\sigma$ is moved farther left, away from the imaginary axis in the $s$ plane, the intervals tend to increase in the parametric space as well, that is, the size of the $Q$ box is increasing. Therefore, by moving the intermediate parameter $\sigma$ continuously, an enveloped region containing a set of overlapped squares, due to the varying bounds, will be obtained, which is shown as the shaded region in the parametric space. This region is important because, for any polynomial with the coefficients in this region, that is, it satisfies Eq. (25), its roots will be in the desired region on the $s$ plane.

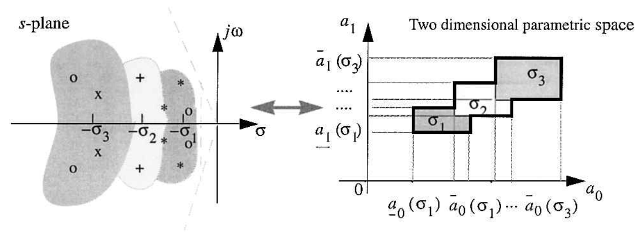



**Fig. 5 Parametric space interpretation of reliable controller design.**

Note that the solution of the reliable state feedback controller $K$ satisfying the inequalities Eq. (27) is not unique. Therefore, depending on different applications, other design requirements can be employed to select a specific $K$ in the feasible solution space.

So far, a reliable control design scheme for MISO systems against actuator failures has been successfully synthesized. The proposed design utilizes a precompensator to modify the input channels to convert the nonuniform actuator redundancy to the uniform one. However, for a general MIMO system, how to choose such a precompensator is still under investigation.

## IV. Illustrative Example

The model used in this example represents a bank-angle control system for a jet transport aircraft flying at the speed of 0.8 Mach and attitude of 40,000 ft. There are two manipulated variables: the aileron and the rudder. The variable being controlled is the bank angle of the aircraft. It is desirable that the bank angle be kept at a desired value even in the presence of failures in the aileron or the rudder.

In a typical flight control system, pilots or automatic flight control systems mainly use the aileron to control the bank angle of the roll motion. However, because there exists dynamic coupling between the yaw and the roll motions, it is also possible to control the bank angle by the rudder. In this respect, the rudder can be considered as a redundant control actuator. It can then be used during an emergency when the aileron control channel malfunctions. This example illustrates how to design a reliable control system with such actuator redundancies so that the stability of the bank-angle control system as well as the steady-state tracking performance can be maintained in the presence of actuator failures.

The model of the nominal system is given as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -0.0558 & -0.9968 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0.0 \\ -3.0500 & 0.3880 & -0.4650 & 0.0 \\ 0.0 & 0.0805 & 1.0000 & 0.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0.0729 & 0.0001 \\ -4.7500 & 1.2300 \\ 1.5300 & 10.6300 \\ 0.0 & 0.0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$y = [0.0 \quad 0.0 \quad 0.0 \quad 1.0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{28}$$

where the states $x_i$, $i = 1, 2, 3, 4$ are the sideslip angle (radian), the yaw rate (radian per second), the roll rate (radian per second), and the bank angle (radian), respectively. The inputs $u_1$ and $u_2$ are the rudder and the aileron, respectively. The output is the bank angle. From the structure of the input matrix of the system, it can be seen that the system redundancy belongs to the nonuniform class. Physically, this means that the aileron and the rudder affect the bank angle of the aircraft differently.

From Eq. (28), the transfer function matrix for this system can be obtained as follows:

$$G(s) = \left( \frac{1.1476s^2 - 2.0036s - 13.7264}{s^4 + 0.6358s^3 + 0.9389s^2 + 0.5116s + 0.0037} \quad \frac{10.7290s^2 + 2.3169s + 10.237}{s^4 + 0.6358s^3 + 0.9389s^2 + 0.5116s + 0.0037} \right) \tag{29}$$

The eigenvalues of the open-loop system are

$$\Lambda = \begin{cases} -0.0329 \pm j0.9467 \\ -0.5627 \\ -0.0073 \end{cases} \tag{30}$$

Based on the design procedure of the precompensator presented in Sec. III.B, a dynamic precompensator is chosen as follows:

$$P(s) = \mathrm{diag}\left( \frac{s^2 + 0.216s + 0.9541}{s^2 + 1.1s + 0.3} \; \frac{s^2 - 1.7459s - 11.9611}{s^2 + 1.1s + 0.3} \right) \tag{31}$$

Noted that the denominator of the precompensator is chosen according to the guidelines specified in Sec. III.B.

With such a precompensator, the augmented system can be represented in state-space form as follows:

$$\dot{\boldsymbol{x}}_a(t) = A_a \boldsymbol{x}_a(t) + B_a \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \boldsymbol{u}(t) + B_a \begin{pmatrix} 1 - l_1 & 0 \\ 0 & 1 - l_2 \end{pmatrix} \boldsymbol{f}_a$$

$$\boldsymbol{y}(t) = C_a \boldsymbol{x}_a(t) \tag{32}$$

where $\boldsymbol{f}_a$ is the unknown actuator offset caused by the failure, and the numerical values for the system matrices are given as

$$A_a = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.0011 & 0.1576 & -0.8481 & -1.7351 & -1.9383 & -1.7358 \end{bmatrix} \tag{33}$$

The input matrix

$$B_a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1.1476 \\ 0 & 0 & 0 & 0 & 0 & 10.7290 \end{pmatrix}^T \tag{34}$$

and the output matrix

$$C_a = [-11.4123 \quad -4.2494 \quad -11.3841 \quad -1.5299 \quad 1 \quad 0] \tag{35}$$

Clearly, Eq. (34) indicates that the redundancies of the system have been converted into a uniform type through the use of the precompensator of Eq. (31).

From the design procedure illustrated in Sec. III.C, the following state feedback gain matrix as well as the $PI$ controller parameters can be obtained. In the design of the controller, the intermediate parameter $\sigma$ is varied from $-0.1$ to $-4.5$ when computing the coefficient interval set in Eq. (25), and the desired pole assignment region is the strip between these two values in the left-half $s$ plane,

$$K = \begin{pmatrix} 4.9197 & 21.6529 & 34.6934 & 30.2875 & 15.9917 & 3.0230 \\ 2.0517 & 6.4617 & 10.8289 & 10.1678 & 4.1151 & 0.7277 \end{pmatrix} \tag{36}$$

$$PI(s) = (-0.0639s - 0.0448)/s \tag{37}$$

In addition, $K$ is obtained from an initially computed feedback gain matrix by minimizing the real part of the largest eigenvalue of the closed-loop system under all actuator failure cases.

The eigenvalues of the closed-loop system under all three modes of operation are illustrated as follows.

1) During normal operation

$$\Lambda = \begin{cases} -4.0120 \\ -3.9878 \\ -3.0001 \\ -1.0782 \\ -0.3468 \\ -0.2937 \pm j0.5001 \end{cases} \tag{38}$$

in which the first three values correspond to the eigenvalues of the precompensator.

2) Under aileron failure

$$\Lambda = \begin{cases} -0.2063 \\ -0.2351 \\ -0.8055 \\ -0.6115 \pm j1.0587 \\ -1.3675 \pm j2.8579 \end{cases} \tag{39}$$

3) Under rudder failure

$$\Lambda = \begin{cases} -0.3405 \\ -0.9712 \\ -2.9570 \\ -0.2087 \pm j0.5478 \\ -2.4285 \pm j3.2267 \end{cases} \tag{40}$$

These eigenvalues clearly indicate that the stability of the system is well maintained in all three system operational modes. Also, for the system configuration shown in Fig. 2, the tracking ability of the system will be maintained with the help of the $PI$ controller in the outer loop. The step responses of the system under all three modes of operation are examined Fig. 6.

When the aileron failure occurs, the system becomes much more sluggish. This is because the aileron is the primary actuator for the bank-angle control. On the other hand, if failure occurs in the rudder, the system experiences slightly more overshoot because during normal operation, the rudder provides additional damping to the closed-loop system. We should emphasize that the steady-state performance of the system is maintained under all three conditions.

The response of the system and the control signals to aileron and rudder are shown in Fig. 7. It would also be interesting to see how failures in the aileron or the rudder affect the bank-angle in the operation of the system. These two situations are shown in Figs. 8a and 9a.

In the simulation, the actuator failure occurs at $t = 30$ s, but it does not manifest itself immediately at the system steady state because such failure results in an offset in the actuator output signal, which acts as a constant disturbance to the system (stuck at a certain position). However, when the input command changes, the failed actuator starts to affect the system output. This can be seen clearly by comparing the three system operations under the three modes after $t = 40$ s. The system response becomes much slower with the

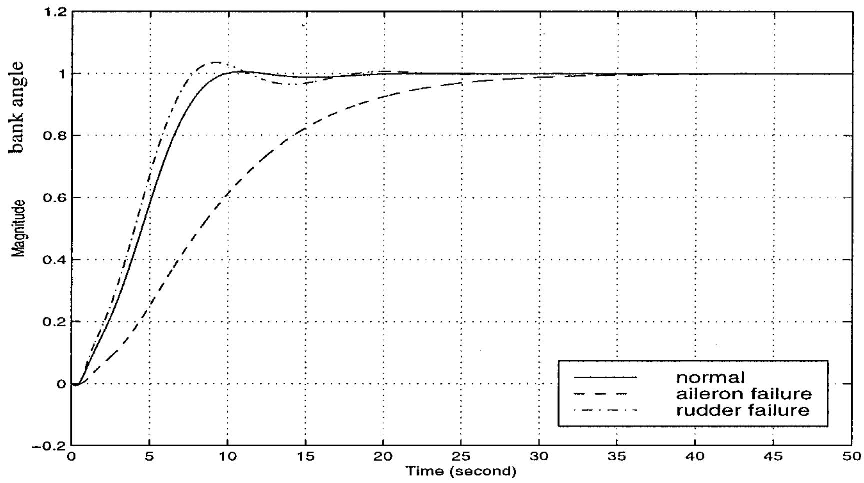
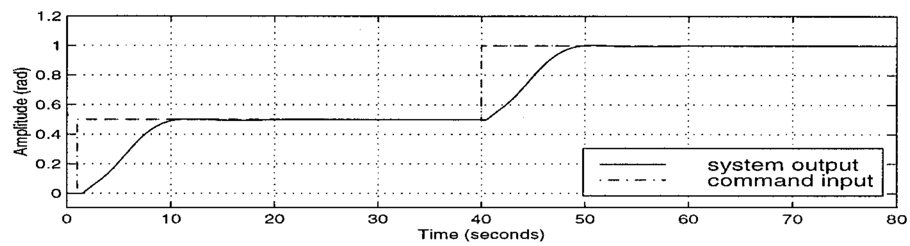
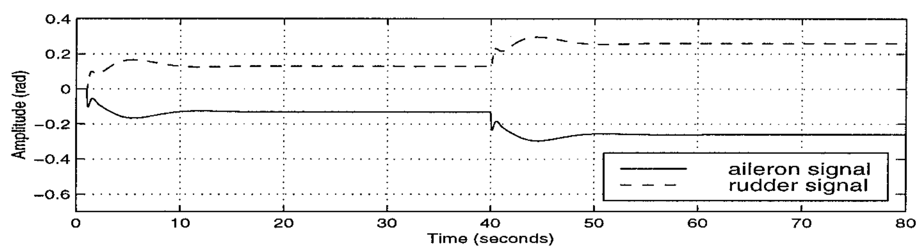



**Fig. 6 Step response of the closed-loop system under three operation modes.**

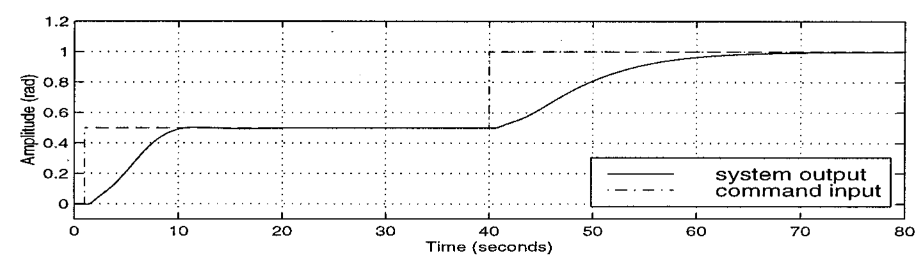



**a) System response**

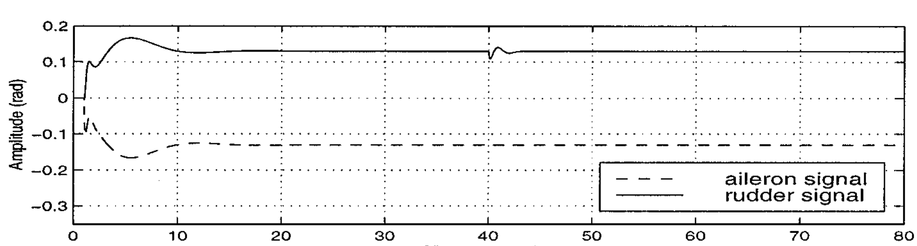



**b) Control signals from aileron and rudder**

**Fig. 7 Normal condition.**

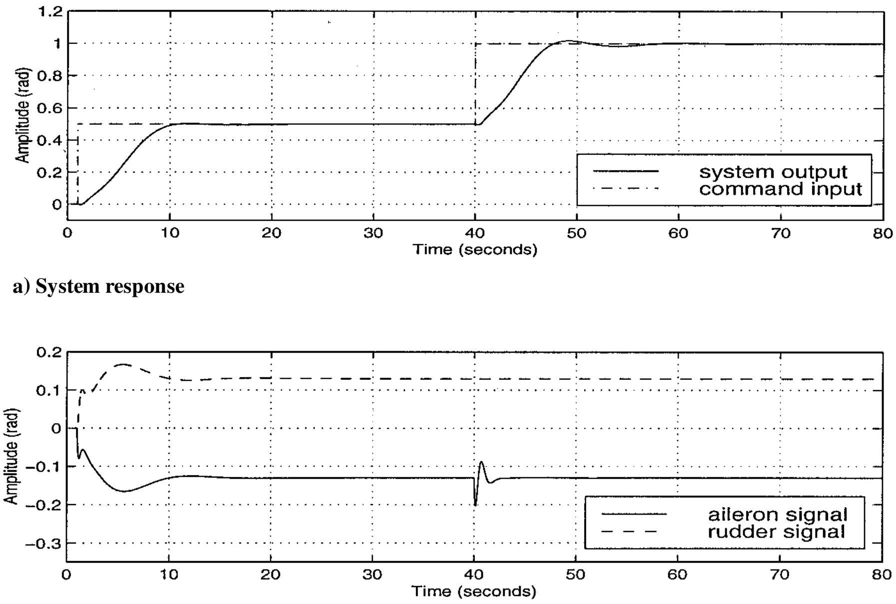



**a) System response**




**b) Control signals from aileron and rudder**

**Fig. 8 Aileron failure.**




a) System response

b) Control signals from aileron and rudder

**Fig. 9 Rudder failure.**

aileron failure (Fig. 8a) and becomes less damped with the rudder failure (Fig. 9a).

To understand better how the reliable controller achieves fault tolerance, and the intricate relationships between the aileron and the rudder in the presence of failures, we have shown the control action from the rudder channel as well as from the aileron channel under all three operational modes in Figs. 7b, 8b, and 9b. Figures 7b–9b clearly show that these two actuators actually counter react against each other. When one actuator fails, it remains at a certain fixed position, and the other actuator changes its control effort immediately to compensate for the failed actuator. This kind of load sharing is carried out internally and automatically with the proposed design scheme.

## V. Conclusions

A novel design technique has been proposed for the synthesis of reliable control systems against actuator failures. Various types of actuator redundancies have been defined with clear physical insight. The structure of the control system consists of an inner loop with a dynamic precompensator, a state-feedback controller, and an outer *PI* control loop. The design involves two main steps: 1) selection of a dynamic precompensator that effectively converts the general actuator redundancy to a more manageable form and 2) design of the state feedback gain matrix and *PI* controller parameters to achieve the stability as well as the steady-state tracking performance in the presence of actuator failures.

The proposed scheme has been applied to the design of an aircraft bank-angle control with the aileron as the primary control actuator and the rudder as its redundant control input. It has been shown that the system can be effectively controlled when both channels are in good operating condition or when either of the two fails.

Even though the concept of redundant actuators have been investigated in a general MIMO framework, the reliable control design procedure is still limited to MISO systems. Research is under way to extend such a design method to the more general case of the MIMO system. Preliminary results indicate that this is a nontrivial problem.

## Appendix: Algorithm to Compute the Nominal Interval Polynomial

In 1991 Soh et al.[30] presented a procedure to determine the allowable perturbation of the coefficients of a polynomial $P(s)$ about their nominal values so that the perturbed polynomial still has all its zeros within a desired region in the open left half-plane.

Assume that an $n$th order polynomial family has the following form:

$$F(s,\varepsilon) = s^n + f_1 s^{n-1} + \cdots + f_n$$

$$t_i - \tau_i\varepsilon \le f_i \le t_i + \mu_i\varepsilon, \qquad i = 1,\ldots,n \tag{A1}$$

where $t_i$ is the $i$th coefficient of the nominal polynomial and $\varepsilon \ge 0$ is the coefficient perturbation. Also, $\mu_i \ge 0$ and $\tau_i \ge 0$ are the two different weights on the perturbation.

The objective is to find the largest positive value of $\varepsilon$, valued $\varepsilon_M$, so that the polynomial family $F(s,\varepsilon)$ has all zeros within the desired region.

By the defining of $F_j(s,\varepsilon)$, $j = 1,\ldots,2^n$, as the vertex polynomial of $F(s,\varepsilon)$, where its coefficients $f_{j,i} = t_i + \mu_i\varepsilon$ or $f_{j,i} = t_i - \tau_i\varepsilon$, $i = 1,\ldots,n$, it can be written as

$$F_j(s,\varepsilon) = P_0(s) + \varepsilon P_j(s) \tag{A2}$$

where

$$P_0(s) = \sum_{i=0}^{n} t_i s^{n-i} \tag{A3}$$

$$P_j(s) = \sum_{i=0}^{n} w_i s^{n-i} \qquad w_i \in \{-\tau_i, \mu_i\} \qquad j = 1,2,\ldots,2^n \tag{A4}$$

*Lemma:* Assume $D$ is a desired region in the open left-half plane, $F(s,\varepsilon)$ has all zeros within the region $D$ if and only if all vertex polynomials $F_j(s,\varepsilon)$, $j = 1,\ldots,2^n$, have all zeros within $D$.

*Proof:* See proof of Lemma 3 in Ref. 30. □

*Theorem:* Suppose $P_0(s)$ has all zeros within the open left-half plane, then the largest positive perturbation $\varepsilon_M$ by which the polynomial family $F(s,\varepsilon)$ has all zeros within the open left-half plane is given by

$$\varepsilon_M = \min\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2^n}\} \tag{A5}$$

where

$$\varepsilon_0 = \begin{cases} t_0/\tau_0 & \tau_0 > 0 \\ +\infty & \tau_0 = 0 \end{cases}$$

$$\varepsilon_i = 1\big/\lambda^{+}_{\max}\left(-R_0^{-1}R_i\right), \qquad i = 1,2,\ldots,2^n \tag{A6}$$

*Proof:* See proof of Theorem 1 in Ref. 30. □

Here $\lambda^{+}_{\max}(\cdot)$ denotes the maximum positive (real) eigenvalue of a matrix. The matrix $R_0$ is associated with the nominal polynomial $P_0(s \cdot je^{j\phi})$, where $\phi$ is shown in Fig. 3. By the writing of $P_0(s \cdot je^{j\phi})$ as

$$P_0(s \cdot je^{j\phi}) = \sum_{i=0}^{n} b_i s^i + j \sum_{i=0}^{n} a_i s^i \tag{A7}$$

the resultant matrix $R_0$ is defined as

$$R_0 = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & a_n & \cdots & \cdots & a_1 & a_0 & \cdots & 0 \\ 0 & b_n & \cdots & \cdots & b_1 & b_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{A8}$$

Similarly, the matrices $R_i$, $i = 1, 2, \ldots, 2^n$, can be obtained from the complex polynomials $P_i(s \cdot je^{j\phi})$, $i = 1, 2, \ldots, 2^n$.

## Acknowledgments

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